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On Self-Similarity of the Pycnocline

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1. THE CONCEPT OF PYCNOCLINE SELF-SIMILARITY

The pycnocline - a layer of strong density gradient adjoining the quasi-homogeneous boundary layer - proves to have a similar form in natural water bodies of different depth and of different overall vertical density difference. A counterpart of the pycnocline can be also found in the atmosphere when convective mixing in the near-surface layer is capped by temperature inversion above. The interface developing in this case at the top of the mixed layer shows a structure similar to the pycnocline below the upper mixed layer in oceans and lakes. A self-similar structure of the density interface is clearly distinguishable in laboratory experiments on entrainment in stratified fluids (Linden 1975, Wyatt 1978). The empirical evidence of the pycnocline self-similarity encouraged a number of researchers to use the pycnocline depth Δh and the density jump across it $\Delta\rho$ as universal length and temperature scales in modelling of vertical density structure of the ocean and the atmosphere. The first ones were apparently Kitaigorodski and Miropolski (1970), who applied the idea of self-similarity to description of the oceanic active layer. Using the above-mentioned scales, the authors introduced dimensionless coordinates

$$\zeta = \frac{z-h(t)}{\Delta h(t)};\ \Theta = \frac{T(z,t)-T_m}{\Delta T} \tag{1}$$

Here, T_m is the temperature of the upper mixed layer of depth $h(t)$ and $\Delta T = T_b - T_m$, is the temperature difference across a thermocline of depth $\Delta h(t)$. An approximated expression for the function $\Theta(\zeta)$ was then found from boundary conditions at the upper and lower thermocline's borders in form of a 4th degree polynomial using Polhausen approach (see fig.1). The idea was subsequently developed using the same solution scheme by other authors (see reviews in Zilitinkevich and Mironov 1992, Tamsalu et al. 1997).

A theoretical explanation for the observed self-similarity was offered for the case of upper mixed layer deepening. The condition dh/dt > 0 allows us to transform the heat transfer equation,

$$\frac{\partial T(t,z)}{\partial t} = -\frac{\partial Q(t,z)}{\partial z} \tag{2}$$

where T is the temperature and Q is the vertical heat flux, by introducing a vertical co-ordinate ζ with an origin moving at the entrainment velocity dh/dt. After the transformation, eq. (2) takes the form:

$$\frac{\partial T(t,\zeta)}{\partial t} - \frac{dh}{dt}\frac{\partial T(t,\zeta)}{\partial \zeta} = -\frac{\partial Q(t,\zeta)}{\partial \zeta} \tag{3}$$

The first term in (3) is of minor magnitude and can be neglected in most of cases (Barenblatt 1978), leaving us with ordinary differential equation with regard to ζ. Some additional assumptions for the heat flux $Q(t,\zeta)$ allow to achieve an analytical solution of (3). Solutions of this kind, usually called the propagating or travelling wave type solutions, are widely used in different branches of mathematical physics. In application to the thermocline problem this analysis was performed independently and simultaneously by Barenblatt (1978) and Turner (1978). Both the authors examined the case of infinitely deep ocean and used a Fickian (gradient) type expression for the turbulent heat flux $Q(t,z) = K_z \, dT(t,z) / dz$ assuming K_z to be a constant or a linear function of the vertical temperature gradient. The uncertainty in the vertical distribution of K_z did not allow them to achieve a solution of the equation (3) corresponding to the self-similar temperature and density profiles observed in reality.

In its turn, the semi-empirical expressions similar to that given in (Kitaigorodski and Miropolski 1970) while reproduce some real situations fairly well, need some additional hypotheses about the $\Theta(\zeta)$ function's behaviour at the lower border of the density interface. It is well known, that underlying stratification can influence the density distribution in the interfacial layer drastically. In terms of the dimensionless co-ordinates (1), this influence can be accounted for by means of the temperature gradient $\Gamma \equiv d\Theta/d\zeta$ at $\zeta = 1$. In application to the oceanic seasonal thermocline, the condition $\Gamma = 0$ is often close to reality and was successfully used by many authors (Kitaigorodski and Miropolski 1970, Arsenyev and Felsenbaum 1977, Mälkki and Tamsalu 1985). However, the condition of zero gradient is not satisfied usually in the atmosphere as well as in shallow lakes. Fedorovich and Mironov (1995) have proposed a semi-empirical expression $\Theta(\zeta,\Gamma)$ for description of the inversion capped convective boundary layer in the atmosphere. The authors have applied the Polhausen approach in couple with an empirical expression for the "shape function" - the integral of the dimensionless temperature Θ across the pycnocline - as a function of background stratification Γ. Extending this approach, we examine a possible representation of the function $\Theta(\zeta,\Gamma)$ not confining ourselves with polynomial representation of Θ.

2. PROPAGATING WAVE TYPE SOLUTION OF THE HEAT TRANSFER EQUATION

We assume the entrainment velocity dh/dt to be much higher than the deepening of the lower pycnocline border. That means $d\Delta h/dt \approx dh/dt$. The situation becomes exactly true in shallow lakes and reservoirs, where the abyssal quiescent layer may not be present so that

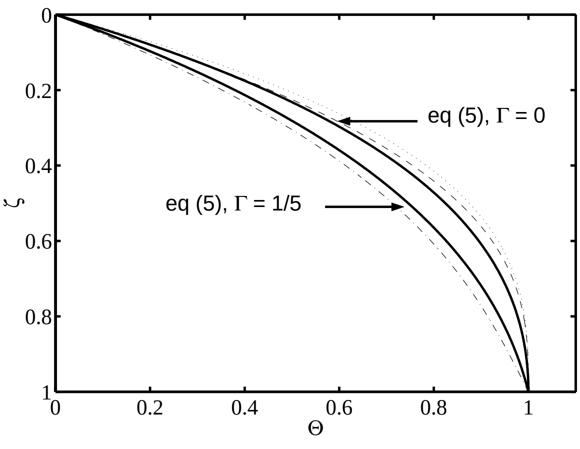


Figure 1 Dimensionless temperature profile; thick solid line – as given by (5); thin lines – previously used approximations: dashed line – $\Gamma = 0$, $\Theta = 8/3\zeta - 2\zeta^2 + \zeta/3$ *(Kitaigorodski and Miropolski 1970); dotted line –* $\Gamma = 0$, $\Theta = 1-(1-\zeta)^3$ *(Arsenyev and Felsenbaum 1977); dash-dotted line –* $\Gamma = 1/5$, $\Theta = 2\zeta - 6/5\zeta^2 + 1/5\zeta^4$ *(Linden, 1975).*

seasonal thermocline extends from the lower edge of the mixed layer down to the basin bottom.

The following conditions have to be satisfied in order to represent the real situation with account of underlying stratification:

$$\Theta = 0 \,\text{at}\, \zeta = 0; \qquad \Theta = 1 \,\text{at}\, \zeta = 1; \qquad \frac{d\Theta}{d\zeta} = \Gamma \,\text{at}\, \zeta = 1; \qquad \int_0^1 \Theta \to 0 \,\text{at}\, \Gamma \to \infty \tag{4}$$

The first two conditions follow directly from the co-ordinates definition (1). The second pair expresses the dependence on underlying stability, where the last condition reproduces the behaviour of the integral shape function in the asymptotic case of two-layered fluid. We search the function $\Theta(\zeta,\Gamma)$ in form $\Theta = \zeta \cdot f(\zeta,\Gamma)$, satisfying the first two conditions automatically. Analysing the second two conditions, the function in question can be written as:

$$\boxed{\Theta = \zeta\, e^{(\zeta-1)(\Gamma-1)}} \tag{5}$$

According to (5), the infinitely increasing underlying stability Γ will lead to degeneracy of the interfacial layer down to the density jump at its lower border. In the second asymptotical case Γ = 0, the expression (5) reduces to:

$$\Theta = \zeta e^{1-\zeta} \tag{6}$$

The shape of dimensionless temperature profile is very close in this case to those found previously using Polhausen method (fig. 1). The case of Γ = 1/5 is also shown in the figure in comparison with the function achieved by (Linden 1975) from laboratory modelling.

The formula (5) can be achieved from the heat transfer equation in the following way. Assuming linear dependence of the water density on temperature, equation (3) can be rewritten in terms of buoyancy $b = -g(\rho-\rho_0)/\rho_0$. Taking into account aforementioned simplifications, equation (3) in the co-ordinates (1) takes the form:

$$(\zeta - 1)\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta} = \frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} \tag{7}$$

Here $\Phi = \langle b'w' \rangle / (\Delta b\, \mathrm{d}h/\mathrm{d}t)$ is the dimensionless buoyancy flux. The buoyancy flux profile corresponding to the solution (6) of the ordinary differential equation (7) can be found as:

$$\Phi = \frac{2-\Gamma}{(\Gamma-1)^2} + \frac{\mathrm{d}\Theta/\mathrm{d}\zeta}{(\Gamma-1)^2}\left[1 + \zeta\Gamma - \zeta - \frac{\zeta(\Gamma^2+\Gamma-2)}{1+\zeta\Gamma-\zeta}\right] \tag{8}$$

which expression reduces in case of Γ = 0 to:

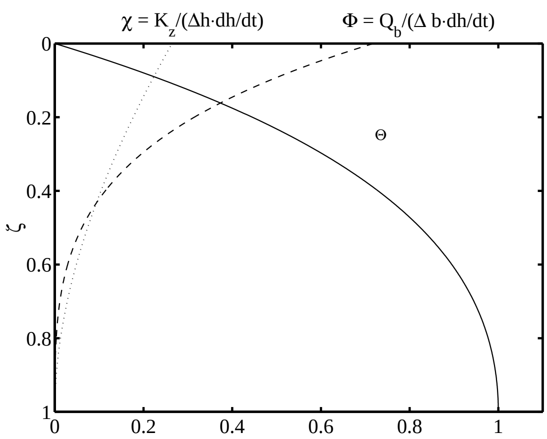


Figure 2 Vertical profiles of dimensionless temperature Θ, heat (or buoyancy) flux Φ and vertical diffusivity χ corresponding the self-similar representation (6).

$$\Phi = 2 - \frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}\left(\zeta^2 + 1\right) / \left(\zeta - 1\right) \tag{9}$$

or, in dimensional co-ordinates:

$$\left\langle b'w' \right\rangle = \left(2\Delta b - \frac{z + \Delta h^2}{z - \Delta h} N^2 \right) \frac{dh}{dt} \tag{10}$$

Here N^2 is the squared Brunt-Väisälä frequency. The expression (10) clarifies the physical meaning of the self-similar buoyancy flux profile. According to it, the turbulent buoyancy flux at the depth z is equal to time changing of potential energy at z due to "compression" of the thermocline with the buoyancy jump across it being constant. In the fig. 2 dimensionless profiles of buoyancy (or temperature), buoyancy flux and diffusion coefficient $K_z = \Phi\left(\mathrm{d}\Theta / \mathrm{d}\zeta\right)^{-1}$, are drawn. The profiles reveal the typical features of turbulent entrainment in stratified fluid and agree with estimations of heat flux profile in the sea thermocline from observations (Tamsalu and Myberg 1998).

3. COMPARISON WITH OBSERVATIONS

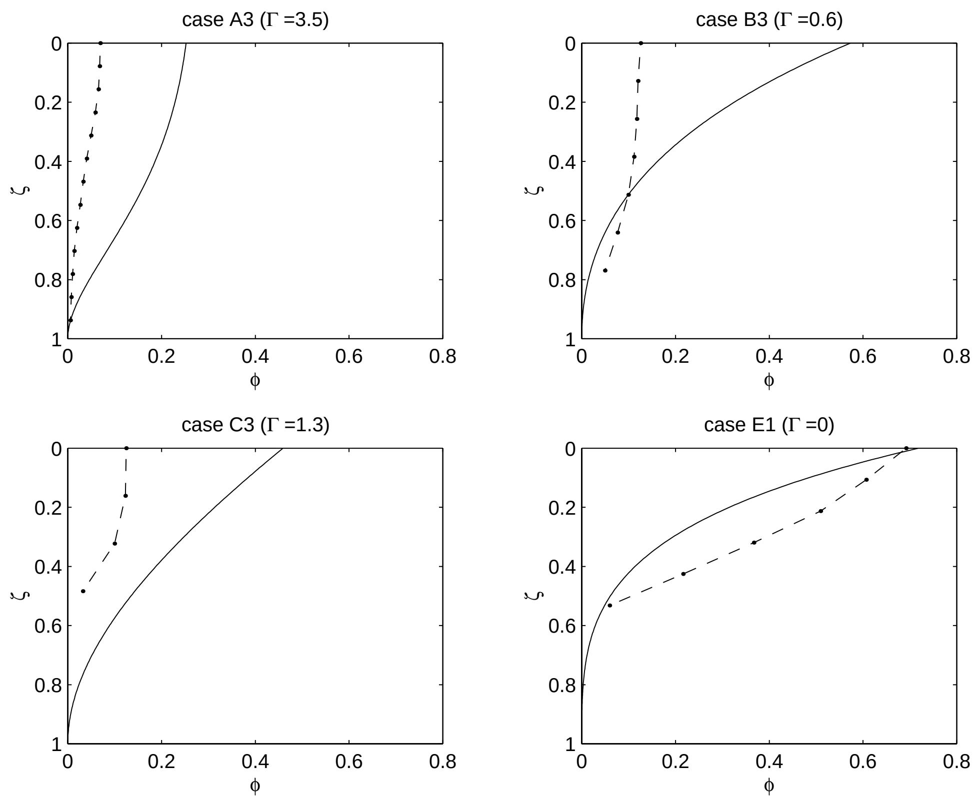
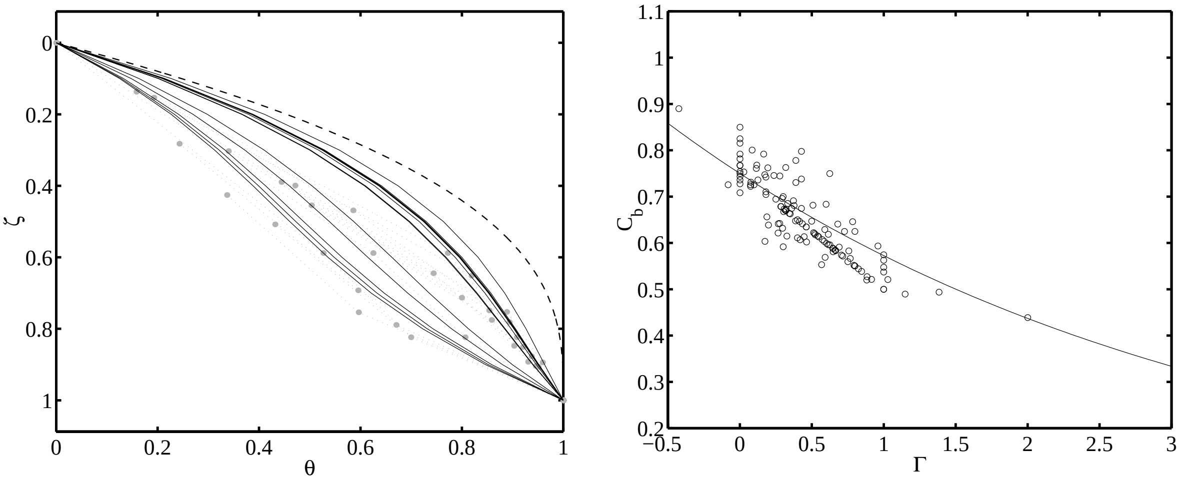


Figure 3 Dimensionless heat flux distribution within the pycnocline on data from laboratory experiments of Deardorff (1979). Dashed lines with dots are measured values, solid lines – bouyancy flux profiles as they found from (8). Four plots present cases with different stratification under the entrainment layer Γ.

The heat flux representation (9) was tested against results of laboratory modelling on entrainment in stratified fluid (Deardorff 1979). The profiles for different underlying stratification are drawn in fig. 3 in terms of dimensionless variables ζ and Φ. The accordance with experimental data is rather quantitative. Nonetheless, in case of neutral stratification under the interfacial layer (case E1 in fig. 3), the solution predicts well the value of buoyancy flux at the top of thermocline as –[exp(1) – 2]*Δ*b***dh/dt*. Uncertainties in heat flux estimation appearing in other cases could result from the fact that the measured values are taken at the initial stage of the experiment, when entrainment process has not stationary nature. In this case, the 1st term in the equation (3) cannot be neglected and the assumption about fixedness of the lower pycnocline border is not valid.

Series of vertical temperature distribution measurements were performed during the summer



(a) (b)

Figure 4. Comparison of the self-similar solution with observations data from the Lake Müggelsee. (a) – Dimensionless temperature profiles for the case of strong bottom temperature gradient; points with dashed lines are measurements, solid lines – as computed with (5); dotted line – asymptotical profile from (6). (b) – Relationship between the bottom temperature gradient and the integral shape factor; circles – measurements, line – theory.

2000 in the Lake Müggelsee located near Berlin, Germany. The observations data were processed in terms of co-ordinates (1). In many typical situations, bottom boundary mixing destroys the vertical temperature gradient below the thermocline and the temperature profile agrees well with that, given by (6). In case of significant stratification at the bottom, the formula (5) reproduces the temperature profile deformation fairly good (fig. 4a). The representativity of the expression (5) is demonstrated by the dependence of the integral shape factor $C_b = \int_0^1 \Theta d\zeta$ on the underlying stratification Γ (fig. 4b), which dependence is described by the solution adequately in the whole range of the Γ variability.

4. CONCLUSION. PRACTICAL APPLICATION IN GEOPHYSICAL MODELS

One of the apparent implementations of the self-similarity solution could be integration of it in one-dimensional bulk-model of stratified shallow lake, accounting water-sediments heat exchange. Practical applications, where computationally cheap parameterised models are favoured over more accurate but more sophisticated models (e.g. second-order turbulence closures), include modelling aquatic ecosystems and numerical weather prediction (NWP). For ecological modelling, a sophisticated physical module is most often not required, and a simple parameterised model would represent the best compromise. For NWP, where rather stringent requirements of computational economy must be met, the concept of self-similarity outlined above is of particular utility. A large number of small lakes that are presently indistinguishable sub-grid scale features will become the resolved-scale features as the horizontal resolution is increased. Such increase is envisaged for most NWP systems in the near future. Then, a physically realistic and at the same time computationally cheap model is required to predict the evolution of the surface temperature of lakes.

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